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MINIMAL FUNCTIONS WITH UNBOUNDED INTEGRAL

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ABSTRACT

We prove theorems which imply the following results. (1) "Most" almost periodic functions $b(t)$ with unbounded integral oscillate in a strong sense. (2) If B is a continuous function on a minimal flow (Ω, R), then either the time averages $(1/t)\int_0^t B(\omega \cdot s)ds$ all converge, or they diverge on a residual set.

§1. Introduction

Let b be an almost periodic (a.p.) function with mean value zero such that $g(t) = \int_0^t b(s)ds$ is unbounded. One might expect that $g(t)$ would oscillate in the sense that $\overline{\lim}_{t\to\infty} g(t) = \infty$, $\underline{\lim}_{t\to\infty} g(t) = -\infty$. This is not the case for all a.p. functions b, as the example $b(t) = \sum_{n=1}^{\infty} (1/n^2)\sin n^2 t$ shows. One can even have $\lim_{t\to\infty} g(t) = \infty$ ([2]).

However, we will show that almost all a.p. functions (with mean value zero and unbounded integral) do have an oscillatory integral: for fixed b, there is a residual subset Ω_0 of the hull (2.1) Ω of b such that, if $b_0 \in \Omega_0$ and $g_0(t) = \int_0^t b_0(s)ds$, then

$$\overline{\lim_{t\to\infty}}\, g_0(t) = \infty, \quad \underline{\lim}_{t\to\infty}\, g_0(t) = -\infty, \quad \overline{\lim_{t\to-\infty}}\, g_0(t) = \infty, \quad \underline{\lim}_{t\to-\infty}\, g_0(t) = -\infty.$$

Moreover, the oscillation is strong in the sense that

$$\underline{\lim}_{n\to\infty} \frac{1}{2n}\, m\{t \in [-n, n] \mid g_0(t) \in I\} = 0$$

for every compact $I \subset \mathbf{R}$ (m = Lebesgue measure on $\mathbf{R}$).

In §3, we will actually prove these statements under weaker assumptions on b,

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namely that (i) the hull Ω of b is minimal (2.2); (ii) for some $b_1 \in \Omega$, and some sequence (t_n) such that $|t_n| \to \infty$, one has $(1/t_n)\int_0^{t_n} b_1(s)ds = 0$; (iii) for some $b_2 \in \Omega$, $\int_0^t b_2(s)ds$ is unbounded. Techniques developed by Sacker and Sell ([11], [12]) for the study of linear skew-product flows (2.2) will be of great importance in the proofs. We will also use an elegant method of Furstenberg, Keynes, and Shapiro ([5], lemma 2.2) to prove a preliminary result (2.7), which reduces to Bohr's theorem ([4], theorem 5.2) when b is a.p.

In §4, the results of §3 are applied to an arbitrary continuous function B defined on a compact metric space Ω, where Ω is the phase space of a minimal flow $(\Omega, \mathbf{R})$. We show that either (i) the time averages $(1/t)\int_0^t B(\omega \cdot s)ds$ converge as $t \to \pm\infty$ for all $\omega \in \Omega$, or (ii) for a residual set of $\omega \in \Omega$, the time averages diverge, both as $t \to \infty$ and as $t \to -\infty$.

§2. Preliminaries

2.1. Definitions. Let $b: \mathbf{R} \to \mathbf{R}$ be a uniformly bounded, uniformly continuous function. For $t \in \mathbf{R}$, let b_t be the *t-translate* of b: $b_t(s) = b(t+s)$ $(s \in \mathbf{R})$. Give $\mathscr{C}(\mathbf{R}, \mathbf{R}) = \{f: \mathbf{R} \to \mathbf{R} \mid f \text{ is continuous}\}$ the topology of uniform convergence on compacta, and let $\Omega = \mathrm{cls}\{b_t \mid t \in \mathbf{R}\} \subset \mathscr{C}(\mathbf{R}, \mathbf{R})$. Then Ω is compact metric. The map $\Phi: \Omega \times \mathbf{R} \to \Omega : (\omega, t) \to \omega_t$ defines a flow ([3]) on Ω; we will denote this flow by $(\Omega, \mathbf{R})$, and write $\omega \cdot t$ for $\Phi(\omega, t)$.

Now let $\omega_0 \equiv b \in \Omega$. Define $B: \Omega \to \mathbf{R}$: $B(\omega) = \omega(0)$. Observe that $B(\omega_0 \cdot t) = \omega_{0,t}(0) = \omega_0(t) = b(t)$. Hence B "extends b from $\{\omega_0 \cdot t \mid t \in \mathbf{R}\}$ to Ω". Clearly $B(\omega \cdot t) = \omega(t)$ $(\omega \in \Omega)$. In what follows, we will refer to $B(\omega \cdot t)$ instead of $\omega(t)$; the reason is that, instead of viewing $(\Omega, \mathbf{R})$ as the hull of some function b, we prefer to think of it as an abstract flow with distinguished continuous function B.

2.2. Definition. A flow $(Y, \mathbf{R})$ with Y compact Hausdorff is *minimal* iff every orbit $\{y \cdot t \mid t \in \mathbf{R}\}$ is dense in Y. Equivalently, $(Y, \mathbf{R})$ is minimal iff the only proper closed invariant subset of Y is the empty set.

2.3. Definitions. Let $(Y, \mathbf{R})$ be a flow with Y compact Hausdorff. A flow $(Y \times \mathbf{R}^n, \mathbf{R})$ is a *linear skew-product flow* (or LSPF) if, for each $(y, x) \in Y \times \mathbf{R}^n$ and each $t \in \mathbf{R}$, one has $(y, x) \cdot t = (y \cdot t, x(t))$, where the map $x \to x(t)$: $\{y\} \times \mathbf{R}^n \to \{y \cdot t\} \times \mathbf{R}^n$ is linear. Suppose $(Y, \mathbf{R})$ is minimal. Say that $\lambda \in \mathbf{R}$ is in the *spectrum* of $(Y \times \mathbf{R}^n, \mathbf{R})$ iff there exists $(y, x) \in Y \times \mathbf{R}^n$ such that $e^{-\lambda t}x(t)$ is bounded $(-\infty < t < \infty)$. See ([11], [12].)

2.4. Theorem. *Let $(Y \times \mathbf{R}^n, \mathbf{R})$ be an LSPF with $(Y, \mathbf{R})$ minimal, and suppose*

β is any number of the form $\lim_{t\to\infty}(1/t_n)\ln\|x(t_n)\|$, where $|t_n|\to\infty$ and $\{(y\cdot t, x(t))\,|\, t\in\mathbf{R}\}$ is some orbit of the LSPF. Then β is in the spectrum of the LSPF.

Theorem 2.4 is an immediate corollary of ([12], theorem 4, p. 185).

Let $(\Omega, \mathbf{R})$ be as in 2.1. In 2.5, we define a flow $(\Sigma, \mathbf{R})$ which will be useful later.

2.5. Definitions. Consider the ordinary differential equations

$$E_\omega : \dot{x} = \begin{pmatrix} 0 & 0 \\ B(\omega\cdot t) & 0 \end{pmatrix} x \qquad (x\in\mathbf{R}^2,\ \omega\in\Omega).$$

Define a flow on $\Omega\times\mathbf{R}^2$ as follows: $(\omega, x_0)\cdot t = (\omega\cdot t, x(t))$, where $x(t)$ is the solution to equation E_ω satisfying $x(0)=x_0$. The flow $(\Omega\times\mathbf{R}^2, \mathbf{R})$ is an LSPF. This LSPF induces a flow $(\Omega\times S^1, R)$, where $S^1\subset\mathbf{R}^2$ is the unit circle, as follows: $(\omega, x_0)\cdot t = (\omega\cdot t, x(t)/\|x(t)\|)$ if $\|x_0\|=1$, where again $x(t)$ solves E_ω with $x(0)=x_0$.

We may describe the flow $(\Omega\times S^1, \mathbf{R})$ more usefully as follows. Let θ be the usual polar coordinate on S^1, with $-\pi\leqq\theta<\pi$. Given $(\omega_0,\theta_0)\in\Omega\times S^1$, define $\theta(t)$ by $(\omega_0,\theta_0)\cdot t\equiv(\omega_0\cdot t,\theta(t))$. By solving equation E_{ω_0}, we find (a) $\theta(t)=\tan^{-1}(\theta_0+\int_0^t B(\omega_0\cdot s)ds)$ if $\theta_0\neq\pm\pi/2$; (b) $\theta(t)\equiv\pm\pi/2$ if $\theta_0=\pm\pi/2$. Hence the set $\Sigma=\Omega\times[-\pi/2,\pi/2]\subset\Omega\times S^1$ is invariant. The flow $(\Sigma,\mathbf{R})$ is the promised flow. Note that, if $\theta_0\neq\pm\pi/2$ and if $\int_0^t B(\omega_0\cdot s)\to+\infty(-\infty)$, then $(\omega_0,\theta_0)\cdot t\to\Omega\times\{\pi/2\}(\Omega\times\{-\pi/2\})$.

2.6. Definitions. Let $b:\mathbf{R}\to\mathbf{R}$ be uniformly bounded and uniformly continuous with hull Ω. Say b is *minimal* if $(\Omega,\mathbf{R})$ is minimal.

2.7. Lemma. *Let $b:\mathbf{R}\to\mathbf{R}$ be minimal, and suppose $\int_0^t b(s)ds$ is bounded as $t\to\infty$ or as $t\to-\infty$. Then there is a continuous $G:\Omega\to\mathbf{R}$ such that $G(\omega\cdot t)-G(\omega)=\int_0^t B(\omega\cdot s)ds$ ($\omega\in\Omega$; B is constructed as in 2.1).*

Proof ([5], lemma 2.2 and [6] §14.11). Define a flow on $\Omega\times\mathbf{R}$ as follows: $(\omega,u)\cdot t=(\omega\cdot t, u+\int_0^t B(\omega\cdot s)ds)$. Also define $T_s:\Omega\times\mathbf{R}\to\Omega\times\mathbf{R}$: $T_s(\omega,u)=(\omega,u+s)$. Let $\omega_0\equiv b\in\Omega$, and define $K_1\subset\Omega\times\mathbf{R}$ to be (a) the ω-limit set ([9]) of the orbit $\{(\omega_0,0)\cdot t\,|\,t\in\mathbf{R}\}$ if $g(t)=\int_0^t B(\omega_0\cdot s)ds$ is bounded as $t\to\infty$; (b) the α-limit set of $\{(\omega_0,0)\cdot t\,|\,t\in\mathbf{R}\}$ if $g(t)$ is bounded as $t\to-\infty$. Then K_1 is compact and invariant, hence contains a nonempty minimal set K ([3]). Since Ω is minimal, the projection of K to Ω is equal to Ω. Suppose that, for some $\omega\in\Omega$, one has (ω,u_1) and $(\omega,u_1+\delta)\in K$. By minimality of K, one has $T_\delta(K)=K$. Hence $T_{n\delta}(K)=K$ for all integers n; hence $\delta=0$. It follows that K covers each point of Ω exactly once. Define $G(\omega)$ to be the unique point of $K \cap \{\omega\} \times \mathbf{R}$; then G satisfies the conditions of 2.7.

2.8. REMARK. If b is almost periodic, then 2.7 is Bohr's theorem ([4], theorem 5.2), since in that case $(\Omega, \mathbf{R})$ is an almost-periodic minimal set ([3]), and $\int_0^t b(s)ds = \int_0^t B(\omega_0 \cdot s)ds = G(\omega_0 \cdot t) - G(\omega_0)$; the last expression is an almost-periodic function of t.

2.9. DEFINITIONS. Let X be compact metric, and let 2^X be the (compact) space of all closed subsets of X with the Hausdorff metric ([1], def. 7.7). If Y is a topological space and $\varphi: Y \to 2^X$ is a map, say that φ is *upper* (*lower*) *semi-continuous* iff for each V open (closed) in X, $\{y \in Y \mid \varphi(y) \cap V \neq \emptyset\}$ is open (closed) in Y.

2.10. THEOREM. *Let X and Y be compact metric, and let $\varphi: Y \to X$ be an upper (lower) semi-continuous map. Then the set of continuity points of φ is residual in Y.*

For a proof of 2.10, see ([1], theorem 7.10) (the proof applies also to upper semi-continuous maps).

§3. Oscillation properties

3.1. NOTATION. Let $b: \mathbf{R} \to \mathbf{R}$ be a minimal function with hull $(\Omega, \mathbf{R})$. Let $B: \Omega \to \mathbf{R}$ and $(\Sigma, \mathbf{R})$ be as in 2.1 and 2.5, respectively. We sometimes write $g_\omega(t)$ for $\int_0^t B(\omega \cdot s)ds$ $(\omega \in \Omega)$.

3.2. ASSUMPTIONS. (1) There exists $\omega_1 \in \Omega$ and a sequence (t_n) with $|t_n| \to \infty$ such that $(1/t_n)\int_0^{t_n} B(\omega_1 \cdot s)ds \to 0$ as $n \to \infty$.

(2) There exists $\omega_2 \in \Omega$ such that $\int_0^t B(\omega_2 \cdot s)ds$ is unbounded.

3.3. THEOREM. *Let m be Lebesgue measure on $\mathbf{R}$ (with $m[0,1] = 1$). Let*

$$\Omega_1 = \left\{\omega \in \Omega \,\middle|\, \varliminf_{n\to\infty} \frac{1}{2n} m\{t \in [-n, n] \mid g_\omega(t) \in I\}\right\} = 0$$

for every compact $I \subset R$. Then Ω_1 is residual in Ω.

PROOF. Let

$$A_{j,k,N} = \left\{\omega \in \Omega \,\middle|\, \frac{1}{2n} m\{t \in [-n, n] \mid g_\omega(t) \in (-k, k)\} \geqq 1/j \text{ for } n \geqq N\right\}.$$

If $\omega_p \to \omega$, then $g_{\omega_p}(t) \to g_\omega(t)$ uniformly on compact subsets of R; hence $\operatorname{cls} A_{j,k,N} \subset A_{j,k+1,N}$. It follows that $\Omega \sim \Omega_1 = \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \operatorname{cls} A_{j,k,N}$.

If the conclusion of 3.3 is false, then some set $A_{j,k,N}$ contains an open set $V \subset \Omega$ (j, k, N are now fixed). Since (Ω, R) is minimal, there exist times $t_1, \cdots, t_p$ such that $\Omega = V \cdot t_1 \cup \cdots \cup V \cdot t_p$ (here $V \cdot t \equiv \{\omega \cdot t \mid \omega \in V\}$). It follows easily that, for some k_0, one has

$$\frac{1}{2n} m\{t \in [-n, n] \mid g_\omega(t) \in (-k_0, k_0)\} \geqq 1/j \quad \text{for all } n \geqq N \text{ and all } \omega \in \Omega.$$

Now consider $g_{\omega_2}(t)$, where ω_2 is given in 3.2(2). Assume $\overline{\lim}_{t\to\infty} g_{\omega_2}(t) = \infty$ (it will be clear that the other possibilities may be handled similarly). Pick times $T_1 < T_2 < \cdots < T_{2j}$ such that $g_{\omega_2}(T_i) = 2ik_0$ $(1 \leqq i \leqq 2j)$. Note that

$$g_{\omega_2 \cdot T_i}(t) = \int_0^t B(\omega_2 \cdot (T_i + s))ds = \int_{T_i}^{T_i+t} B(\omega_2 \cdot s)ds = g_{\omega_2}(T_i + t) - g_{\omega_2}(T_i).$$

For $n \geqq N$, one has

$$\frac{1}{2n} m\{t \in [-n, n] \mid g_{\omega_2 \cdot T_i}(t) \in (-k_0, k_0)\} \geqq 1/j;$$

hence, if $n \geqq N$, then $g_{\omega_2}(s)$ is in the open interval $((2i-1)k_0, (2i+1)k_0)$ for a set $S_i \subset [T_i - n, T_i + n]$ satisfying $(1/2n)m(S_i) \geqq 1/j$.

Now choose $n \geqq \max(N, 2j \cdot T_{2j})$. Then $(1/2n)m(S_i \cap [-n, n]) \geqq 1/2j$ $(1 \leqq i \leqq 2j)$. Since

$$[-n, n] \supset \bigcup_{i=1}^{2j} (S_i \cap [-n, n]) \cup \{s \in [-n, n] \mid g_{\omega_2}(s) \in (-k_0, k_0)\},$$

and since the sets on the right hand side are pairwise disjoint, we obtain $1 = (1/2n)m[-n, n] \geqq (2j)(1/2j) + 1/j > 1$, a contradiction.

3.4. Definition. Let $\Omega_2 = \{\omega \in \Omega \mid \overline{\lim}_{t\to\infty} g_\omega(t) = \infty,\ \underline{\lim}_{t\to\infty} g_\omega(t) = -\infty,\ \overline{\lim}_{t\to-\infty} g_\omega(t) = \infty,\ \underline{\lim}_{t\to-\infty} g_\omega(t) = -\infty\}$.

3.5. Lemma ([12], p. 204, prob. 7). (1) *There exists* $\omega_3 \in \Omega$ *and a real* M *such that* $g_{\omega_3}(t) \leqq M$ $(-\infty < t < \infty)$.

(2) *There exists* $\omega_4 \in \Omega$ *and a real* M *such that* $g_{\omega_4}(t) \geqq M$ $(-\infty < t < \infty)$.

Proof. Consider the linear skew-product flow (2.3) $(\Omega \times \mathbf{R}, \mathbf{R})$ defined as follows: $(\omega, x) \cdot t = (\omega \cdot t, x_\omega(t))$, where $x_\omega(t) = x \cdot \exp[\int_0^t B(\omega \cdot s)ds]$. By 2.4, any number of the form $\lim_{n\to\infty}(1/t_n)\ln|x_\omega(t_n)|$ is in the spectrum (2.2) of $(\Omega \times \mathbf{R}, \mathbf{R})$, where $|1/t_n| \to \infty$. By 3.2(1), zero is in the spectrum of $(\Omega \times \mathbf{R}, \mathbf{R})$ so there exists $\omega_3 \in \Omega$ such that $\exp[\int_0^t B(\omega_3 \cdot s)ds]$ is bounded. This proves (1). To prove (2), replace B by $-B$ in the above argument.

Consider the flow (Σ, R) of 2.5; recall $\Sigma = \Omega \times [-\pi/2, \pi/2]$. Let $K_n = \Omega \times [-\pi/2 + 1/n, \pi/2] \subset \Sigma$ $(n = 1, 2, \cdots)$. Let

$$V_n = \{(\omega, \theta) \in \Sigma \mid (\omega, \theta) \cdot t \in K_n \text{ for all } t \geqq 0\}$$

$$= \left\{(\omega, \theta) \in \Sigma \,\middle|\, \theta + \int_0^t B(\omega \cdot s) \geqq \tan^{-1}(-\pi/2 + 1/n) \text{ for all } t \geqq 0\right\}.$$

Then V_n is compact, and it is positively invariant (i.e., $(\omega, \theta) \cdot t \in V_n$ whenever $(\omega, \theta) \in V_n$ and $t \geqq 0$). Note $\Omega \times \{\pi/2\} \subset V_n$ for all $n \geqq 1$.

3.6. Lemma. *Let*

$$W_n = \{\omega \in \Omega \mid \textit{there exists } \theta, -\pi/2 < \theta < \pi/2, \textit{ with } (\omega, \theta) \in V_n\}.$$

Then W_n is of first category in Ω.

Proof. Let $Z \equiv 2^{[-\pi/2, \pi/2]}$ be the set of closed subsets of $[-\pi/2, \pi/2]$ with the Hausdorff metric (see 2.9). For fixed n, define $\varphi : \Omega \to Z : \omega \to V_n \cap (\{\omega\} \times [-\pi/2, \pi/2])$. Then φ is lower semi-continuous (2.9), hence has a residual set of continuity points (2.10). Let $\bar{\omega}$ be a continuity point of φ. To prove 3.6, it is sufficient to prove that $\bar{\omega} \not\in W_n$.

So, suppose $\bar{\omega} \in W_n$, and let $(\bar{\omega}, \theta_0) \in V_n$, $-\pi/2 < \theta_0 < \pi/2$. By continuity at $\bar{\omega}$, there exists a neighborhood $\mathcal{O}$ of $\bar{\omega}$ such that, for each $\omega \in \mathcal{O}$, there exists $\theta_\omega \in (-\pi/2, \pi/2)$ with $(\omega, \theta_\omega) \in V_n$. Since $(\Omega, \mathbf{R})$ is minimal, there exist finitely many positive times $t_1, \cdots, t_n$ such that $\mathcal{O} \cdot t_1 \cup \cdots \cup \mathcal{O} \cdot t_n = \Omega$. Since V_n is positively invariant, it follows that, for *every* $\omega \in \Omega$, there exists $\theta_\omega \in (-\pi/2, \pi/2)$ such that $(\omega, \theta_\omega) \in V_n$.

Now, from the definition of $(\Sigma, \mathbf{R})$ the last statement implies that, for each ω, there exists $M_\omega \in \mathbf{R}$ such that $\int_0^t B(\omega_3 \cdot s) ds \geqq M_\omega$ for all $t \geqq 0$. But, by 3.5(1), there exists ω_3 and M such that $\int_0^t B(\omega_3 \cdot s) ds \leqq M$ $(-\infty < t < \infty)$. By 2.7, $\int_0^t B(\omega \cdot s) ds$ is bounded on $-\infty < t < \infty$ for *all* ω, contradicting 3.2(3). This completes the proof of 3.6.

3.7. Theorem. *Let Ω_2 be as in* 3.4. *Then Ω_2 is residual in Ω.*

Proof. First let $W_a = \bigcup_{n=1}^\infty W_n$, where W_n is as in 3.6. Then $\Omega \sim W_a = \{\omega \in \Omega \mid \underline{\lim}_{t\to\infty} g_\omega(t) = -\infty\}$, and by 3.6, $\Omega \sim W_a$ is residual in Ω.

Replace K_n by $\Omega \times [-\pi/2, \pi/2 - 1/n]$ $(n = 1, 2, \cdots)$ in the proof of 3.6. One concludes that $\{\omega \in \Omega \mid \overline{\lim}_{t\to\infty} g_\omega(t) = \infty\}$ is residual in Ω. Replacing "$t \geqq 0$" by "$t \leqq 0$" in the definition of V_n, and proceeding as above, one also shows that $\{\omega \in \Omega \mid \overline{\lim}_{t\to-\infty} g_\omega(t) = \infty \text{ and } \underline{\lim}_{t\to-\infty} g_\omega(t) = -\infty\}$ is residual in Ω. This completes the proof.

Combining 3.3 and 3.7, we obtain

3.8. THEOREM. *$\Omega_0 = \Omega_1 \cap \Omega_2$ is residual in Ω.*

3.9. REMARKS. (1) If $b: \mathbf{R} \to \mathbf{R}$ is an almost periodic function with mean value zero and unbounded integral, then 3.2(1) and 3.2(2) are satisfied. Also, the hull of b is minimal (in fact, minimal and almost periodic ([3])).

(2) In ([7]), it is shown (using an example of ([8])) that there exist almost periodic functions b with mean value zero and unbounded integral such that $\lim_{n\to\infty}(1/2n)m\{t \in [-n, n] \mid \int_0^t b(s) \in I\}$ exists and is positive for some compact I. On the other hand, it is also shown in [7] that there exist almost periodic functions b with mean value zero such that $\lim_{n\to\infty}(1/2n)m\{t \in [-n, n] \mid g_\omega(t) \in I\} = 0$ for *all* ω, I.

(3) Let Ω be compact metric, $T: \Omega \to \Omega$ a homeomorphism, and suppose the integer flow (Ω, T) is minimal. Let $B: \Omega \to \mathbf{R}$ be continuous. Define a homeomorphism $\tilde{T}$ of $\Sigma = \Omega \times [-\pi/2, \pi/2]$ as follows: $\tilde{T}(\omega, \theta) = (T(\omega), \theta + B(\omega))$. For positive integers k, define $g_\omega(k) = \Sigma_{i=0}^{k-1} B(T^i(\omega))$; for negative k, define $g_\omega(k) = \Sigma_{i=0}^{k-1} B(T^{-i}(\omega))$. Then all methods used above apply with "k" in place of "t" and "$g_\omega(k)$" in place of "$g_\omega(t)$". We conclude that one of the following holds: either (a) the sums $\Sigma_{i=0}^{k-1} B(T^i(\omega))$ and $\Sigma_{i=0}^{k-1} B(T^{-i}(\omega))$ are bounded, uniformly in k and ω (2.7), or (b) there is a residual set $\Omega_0 \subset \Omega$ such that $\omega \in \Omega_0$ implies (i) $\underline{\lim}_{n\to\infty}(1/2n)\mathrm{card}\{k \in [-n, n] \mid g_\omega(k) \in I\} = 0$ for all compact $I \subset \mathbf{R}$, and (ii) $g_\omega(k)$ oscillates in the sense of 3.4 (i.e., $\overline{\lim}_{k\to\infty} g_\omega(k) = \infty$, etc.).

§4. Time averages

4.1. NOTATION. We retain the notation of 3.1. Let $E(\Omega)$ be the set of (Radon) probability measures on Ω which are ergodic ([9]) with respect to the flow $(\Omega, \mathbf{R})$.

The following is well-known (see, e.g., [10]).

4.2. THEOREM. *Let $M(\Omega)$ be the compact convex set of (Radon) probability measures on Ω which are invariant with respect to $(\Omega, \mathbf{R})$. Then $M(\Omega)$ is the closed convex hull of $E(\Omega)$.*

Let $B: \Omega \to \mathbf{R}$ be as in 3.1 (note that any continuous function B on Ω is "induced" by a function $b: \mathbf{R} \to \mathbf{R}$; simply define $b(t) \equiv B(\omega_0 \cdot t)$ for some $\omega_0 \in \Omega$).

4.3. THEOREM. (1) *Suppose that, for some fixed $b_0 \in \mathbf{R}$, $\int_\Omega B(\omega)d\mu(\omega) = b_0$ for all $\mu \in E(\Omega)$. Then*

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t B(\omega\cdot s)ds = b_0 \qquad \textit{for all } \omega\in\Omega.$$

(2) *Suppose* $b_1 = \int_\Omega B(\omega)d\mu_1(\omega) \neq \int_\Omega B(\omega)d\mu_2(\omega) = b_2$ *for* $\mu_1, \mu_2 \in E(\Omega)$. *Then*

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\int_0^t B(\omega\cdot s)ds \neq \underline{\lim_{t\to\infty}}\frac{1}{t}\int_0^t B(\omega\cdot s)ds$$

and

$$\overline{\lim_{t\to-\infty}}\frac{1}{t}\int_0^t B(\omega\cdot s)ds \neq \underline{\lim_{t\to-\infty}}\frac{1}{t}\int_0^t B(\omega\cdot s)ds$$

for a residual set of $\omega\in\Omega$.

PROOF. (1) We use the method of ([9], p. 494). Suppose there is a sequence (t_n) with $|t_n|\to\infty$ and a point $\omega_0\in\Omega$ such that

$$\underline{\lim_{n\to\infty}}\frac{1}{t_n}\int_0^{t_n} B(\omega_0\cdot s)ds \neq b_0.$$

We may choose a subsequence (r_n) of (t_n) such that

$$\lim_{n\to\infty}\frac{1}{r_n}\int_0^{r_n} F(\omega_0\cdot s)ds \equiv \bar\mu(F)$$

exists for all continuous functions $F:\Omega\to\mathbf{R}$; moreover, the map $F\to\bar\mu(F)$ defines (using the Riesz theorem) an *invariant* measure $\bar\mu$ on Ω (see [9], pp. 494–495). Then $\int_\Omega B(\omega)d\bar\mu(\omega)\neq b_0$. However, by 4.2 and our assumption on b, $\int_\Omega B(\omega)d\bar\mu(\omega)$ must equal b_0. This contradiction proves (1).

(2) Consider the functions $B_1(\omega) = B(\omega) - b_1$ and $B_2(\omega) = B(\omega) - b_2$. Clearly 3.2(1) holds for B_1 and B_2. If 3.2(2) did not hold for, say, B_1, then 2.7 would imply that $\int_0^t B(\omega\cdot s)ds = b_1 t + G_1(\omega\cdot t) - G_1(\omega)$, where $G_1:\Omega\to\mathbf{R}$ is continuous. But the Birkhoff ergodic theorem ([9]) implies that, for μ_2-a.a. ω,

$$\lim_{|t|\to\infty}\frac{1}{t}\int_0^t B(\omega\cdot s)ds = b_2.$$

This contradicts the previous statement. So 3.2(1) and 3.2(2) hold for B_1 and B_2.

Now apply 3.7 to B_1 and B_2; one obtains a residual $\Omega_3\subset\Omega$ such that, if $\omega\in\Omega_3$, (i) there is a sequence $(t_n)\to\infty$ with $\int_0^{t_n} B_1(\omega\cdot s)ds = 0$ (which implies

$\int_0^{t_n} B(\omega \cdot s)ds = b_1 \cdot t_n$); (ii) there is a sequence $(t_m) \to \infty$ with $\int_0^{t_m} B(\omega \cdot s)ds = b_2 \cdot t_m$. Also, there are analogous sequences $(t'_n) \to -\infty$, $(t'_m) \to -\infty$. Each $\omega \in \Omega_3$ satisfies the conditions of 4.3(2).

4.4. Remarks. (1) Theorem 4.3 states that, if $(\Omega, \mathbf{R})$ is minimal, and if some time average $(1/t)\int_0^t B(\omega_0 \cdot s)ds$ diverges as $t \to \infty$ (or as $t \to -\infty$), then the time averages diverge for a residual set of $\omega \in \Omega$.

(2) As in 3.9(3), we can replace integrals by sums if a minimal discrete flow (Ω, T) is given. We may interpret 4.3 as follows: if the Césaro sums $(1/k)\Sigma_{i=0}^{k-1} B(T^i(\omega))$ diverge at a single point, they diverge on a residual set; the same statement holds for $(1/k)\Sigma_{i=0}^{k-1} B(T^{-i}(\omega))$.

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